

# *Linear Algebra with Applications* *Third Edition*

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A zero matrix (Definition 7) acts as an additive identity in the set of matrices of the same size. What matrix is a multiplicative identity?

If the matrix  $\mathbf{I}$  is a multiplicative identity,  $\mathbf{IB} = \mathbf{B}$ . If  $\mathbf{B}$  is  $m \times n$ , then  $\mathbf{I}$  must have  $m$  columns for multiplication to be defined, and  $\mathbf{I}$  must have  $m$  rows for the product  $\mathbf{IB}$  to have  $m$  rows. Thus the condition  $\mathbf{IB} = \mathbf{B}$  forces  $\mathbf{I}$  to be a square matrix.

**Example 19**

Let  $\mathbf{B}$  be a  $2 \times 3$  matrix, and  $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . Then

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

so that  $\mathbf{I}_2 \mathbf{B} = \mathbf{B}$  for every  $2 \times 3$  matrix  $\mathbf{B}$ . The product  $\mathbf{BI}_2$  is not defined since  $\mathbf{B}$  has three columns. However,

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

**Identity Matrix****Definition 11**

An  $n \times n$  matrix with the property that  $a_{ii} = 1$  and  $a_{ij} = 0$  for  $i \neq j$  is called an **identity matrix**,  $\mathbf{I}_n$ . When the context makes the size of  $\mathbf{I}$  clear, the subscript is omitted.

A zero matrix, like the number 0, also has special properties related to multiplication. For example, if defined, a product in which one factor is a zero matrix always produces a zero matrix.

**Example 20**

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

In this case  $\mathbf{0}_{2 \times 3} \mathbf{A}_{3 \times 2} = \mathbf{0}_{2 \times 2}$ . The matrix  $\mathbf{A}_{3 \times 2} \mathbf{0}_{2 \times 3}$  can also be calculated and the result is the zero matrix,  $\mathbf{0}_{3 \times 3}$ . ■

One property of the real number 0 does not carry over into the matrix setting. If the product of two real numbers is 0, we can conclude that at least one of the factors is 0; that is,  $ab = 0$  implies either  $a = 0$  or  $b = 0$ . There is no such property in matrix multiplication.

T.S. Blyth and E.F. Robertson

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# Basic Linear Algebra



Springer

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Our next result is the multiplicative analogue of Theorem 1.2, but the reader should note that it applies only in the case of square matrices.

### Theorem 1.8

There is a unique  $n \times n$  matrix  $M$  with the property that, for every  $n \times n$  matrix  $A$ ,  $AM = A = MA$ .

### Proof

Consider the  $n \times n$  matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

More precisely, if we define the **Kronecker symbol**  $\delta_{ij}$  by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise,} \end{cases}$$

then we have  $M = [\delta_{ij}]_{n \times n}$ . If  $A = [a_{ij}]_{n \times n}$  then  $[AM]_{ij} = \sum_{k=1}^n a_{ik} \delta_{kj} = a_{ij}$ , the last equality following from the fact that every term in the summation is 0 except that in which  $k = j$ , and this term is  $a_{ij}1 = a_{ij}$ . We deduce, therefore, that  $AM = A$ . Similarly, we can show that  $MA = A$ . This then establishes the existence of a matrix  $M$  with the stated property.

To show that such a matrix  $M$  is unique, suppose that  $P$  is also an  $n \times n$  matrix such that  $AP = A = PA$  for every  $n \times n$  matrix  $A$ . Then in particular we have  $MP = M = PM$ . But, by the same property for  $M$ , we have  $PM = P = MP$ . Thus we see that  $P = M$ .  $\square$

### Definition

The unique matrix  $M$  described in Theorem 1.8 is called the  $n \times n$  **identity matrix** and will be denoted by  $I_n$ .

Note that  $I_n$  has all of its 'diagonal' entries equal to 1 and all other entries 0. This is a special case of the following important type of square matrix.

### Definition

A square matrix  $D = [d_{ij}]_{n \times n}$  is said to be **diagonal** if  $d_{ij} = 0$  whenever  $i \neq j$ . Less formally,  $D$  is diagonal when all the entries off the main diagonal are 0.

## EXERCISES

1.19 If  $A$  and  $B$  are  $n \times n$  diagonal matrices prove that so also is  $AB$ .

# A P R I M E R O N LINEAR ALGEBRA

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Perhaps more interesting are the products

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0.$$

Note several things about the product of matrices:

1. If  $A, B$  are in  $M_2(\mathbb{R})$ , then  $AB$  is also in  $M_2(\mathbb{R})$ .
2. In  $M_2(\mathbb{R})$ , it is possible that  $AB = 0$  with  $A \neq 0$  and  $B \neq 0$ .
3. In  $M_2(\mathbb{R})$ , it is possible that  $AB \neq BA$ .

These last two behaviors both run counter to our prior experience with number systems, where we know that

- 2'. In  $\mathbb{R}$ ,  $ab = 0$  if and only if  $a = 0$  or  $b = 0$ .
- 3'. In  $\mathbb{R}$ ,  $ab = ba$  for all  $a$  and  $b$ .

Here (2') is, in effect, the *cancellation law of multiplication* for real numbers:

*If  $ab = 0$  and  $a \neq 0$ , then  $b = 0$ .*

Thus (2) says that

*The cancellation law of multiplication does not hold in  $M_2(\mathbb{R})$ .*

At the same time, cancellation is possible in  $M_2(\mathbb{R})$  under certain circumstances, as we observe in Problem 10. Similarly, (3') says that real numbers *commute* under multiplication. Thus (3) says that

*Matrices in  $M_2(\mathbb{R})$  do not necessarily commute under multiplication.*

Matrices in  $M_2(\mathbb{R})$  satisfy the *associative law* that

$$(AB)C = A(BC)$$

as you can see by multiplying out the expressions on both sides of the equation. We leave this as an easy, though tedious exercise.

A matrix that plays a very special role in multiplication is the matrix

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . This matrix is called the *identity matrix*, because it has the following properties:

$$\begin{aligned} AI &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a \cdot 1 + b \cdot 0 & a \cdot 0 + b \cdot 1 \\ c \cdot 1 + d \cdot 0 & c \cdot 0 + d \cdot 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A. \end{aligned}$$

Similarly,

$$\begin{aligned} IA &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot a + 0 \cdot c & 1 \cdot b + 0 \cdot d \\ 0 \cdot a + 1 \cdot c & 0 \cdot b + 1 \cdot d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A. \end{aligned}$$

Thus, multiplying any matrix  $A$  by  $I$  on either side does not change  $A$  at all. In other words, the matrix  $I$  in  $M_2(\mathbb{R})$  behaves very much like the number 1 does in  $\mathbb{R}$  when one multiplies.

For every nonzero real number  $a$  we can find a real number, written as  $a^{-1} = 1/a$ , such that  $aa^{-1} = 1$ . Is something similar true here in the system  $M_2(\mathbb{R})$ ? The answer is "no." More specifically, we cannot find, for every nonzero matrix  $A$ , a matrix  $A^{-1}$  such that  $AA^{-1} = I$ .

Consider, for instance, the matrix  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ . Can we find a matrix

$$B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

such that  $AB = I$ ? Let's see what is needed. What we require for  $AB = I$  is that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} e & f \\ 0 & 0 \end{bmatrix}.$$

This would require that  $e = 1$ ,  $f = 0$  and the absurdity that  $1 = 0$ . So no such  $B$  exists for this particular  $A$ . Let's try another one, the matrix

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ , where the result is quite different. Again we ask whether

we can find a matrix  $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$  such that  $AB = I$ . Again, let's see

what is needed. What is required in this case is that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} e+g & f+h \\ g & h \end{bmatrix}.$$

This requires that  $g = 0$ ,  $h = 1$ ,  $e = 1$ ,  $f = -1$ , so that the matrix

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

does satisfy  $AB = I$ . Moreover, this matrix  $B$  also satisfies the equation  $BA = I$ , which you can easily verify.

We have seen that for some matrices  $A$  we can find a matrix  $B$  such that  $AB = BA = I$ , and that for some  $A$  no such  $B$  can be found. We single out these "good" ones in our growing terminology.

**Definition.** A matrix  $A$  is said to be *invertible* if we can find a matrix  $B$  such that  $AB = BA = I$ .

A matrix  $A$  which is not invertible is called *singular*. If  $A$  is invertible, we claim that the  $B$  above is *unique*. What exactly does this mean? It means merely that if  $AC = CA = I$  for a (possibly) different matrix  $C$ , then  $B = C$ . To see that  $AB = BA = I$  and  $AC = CA = I$  imply that  $B = C$ , just equate  $AB = AC$  and cancel  $A$  by multiplying each side on the left by  $B$ . We leave the details as an exercise. You will have to use the associative law for this (see Problem 3).

If  $A$  is invertible and  $AB = BA = I$  as above, we call  $B$  the *inverse* of  $A$  and in analogy to what we do in the real numbers, we write  $B$  as  $A^{-1}$ . We stress again that *not all* matrices are invertible. In a short while we shall see how the entries of  $A$  determine whether or not  $A$  is invertible.

We now come to two particular, easy-looking classes of matrices.

**Definition.** The matrix  $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$  is called a *diagonal matrix*.

**Definition.** The matrix  $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$  is called a *scalar matrix*.

So a matrix is diagonal if its off-diagonal entries are 0. And it is a scalar matrix if, in addition, the diagonal entries are equal.

Let  $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$  be a scalar matrix. Then, if we multiply any

# Basic Concepts of Linear Algebra

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in general, one may *not* cancel the matrix  $C$  in an equation  $AC = BC$ .)

### 1-3 Elementary Matrices

#### Objective

Introduce the concept of an elementary matrix of first, second, and third kinds.

This section will be devoted to the introduction of the *identity matrix* and the *elementary matrices* associated with it. These matrices play a significant role in applications discussed later in this chapter.

Let  $A$  be an  $n \times n$  matrix. We say that  $A$  is a **square matrix of order  $n$** . We sometimes denote such a matrix by  $A_n$  to emphasize the dimension. The entries  $a_{ii}$  ( $i = 1, 2, \dots, n$ ) are called the **elements of the main diagonal** of the matrix  $A_n$ .

#### Definition 1-6

The identity matrix of order  $n$ .

The matrix  $I_n$  defined by

$$I_n = (\delta_{ij})$$

where  $\delta_{ij}$  is the Kronecker delta, is called the **identity matrix of order  $n$** .

We may describe  $I_n$  by saying that all the entries of its main diagonal are 1 while all other entries of  $I_n$  are 0. Thus,

$$I_1 = [1], I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and so on.

Let  $A$  be a  $2 \times 3$  matrix. Then we have

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

and

$$I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Thus, the effect of multiplying the matrix  $A$  by an identity matrix  $I$  (of proper order) from the left or from the right is to leave  $A$  unchanged.

It follows from the definition of matrix multiplication and the definition of the identity matrix  $I$  that we have

$$BI_n = B$$

and

$$I_m B = B$$

for every  $m \times n$  matrix  $B$ .

In the following examples we shall examine the effect of multiplying a given matrix  $A$  (from the left) by special matrices derived from the identity matrix  $I$ .

**Example 1** Let

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Find  $EA$ .

**Solution**

$$EA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 2 & 5 \\ 1 & 4 \end{bmatrix}.$$

A close examination of the resulting matrix shows that one gets

the matrix  $\begin{bmatrix} 3 & 6 \\ 2 & 5 \\ 1 & 4 \end{bmatrix}$  from the matrix  $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$  by interchanging the first

and the third rows. Also, one can easily see that interchanging the first and the third rows of  $I_3$  produces the matrix  $E$ . Thus, we may think of  $E$  as a "row changer" when it multiplies a matrix  $A$  from the left. This leads to the following definition.

**Definition 1-7** We call  $E$  an **elementary matrix of the first kind** if  $E$  is obtained from the identity matrix  $I_n$  by interchanging any two of its rows.

Elementary matrix of the first kind

An important property of an elementary matrix  $E$  of the first kind is illustrated in the next example.

**Example 2** Let

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

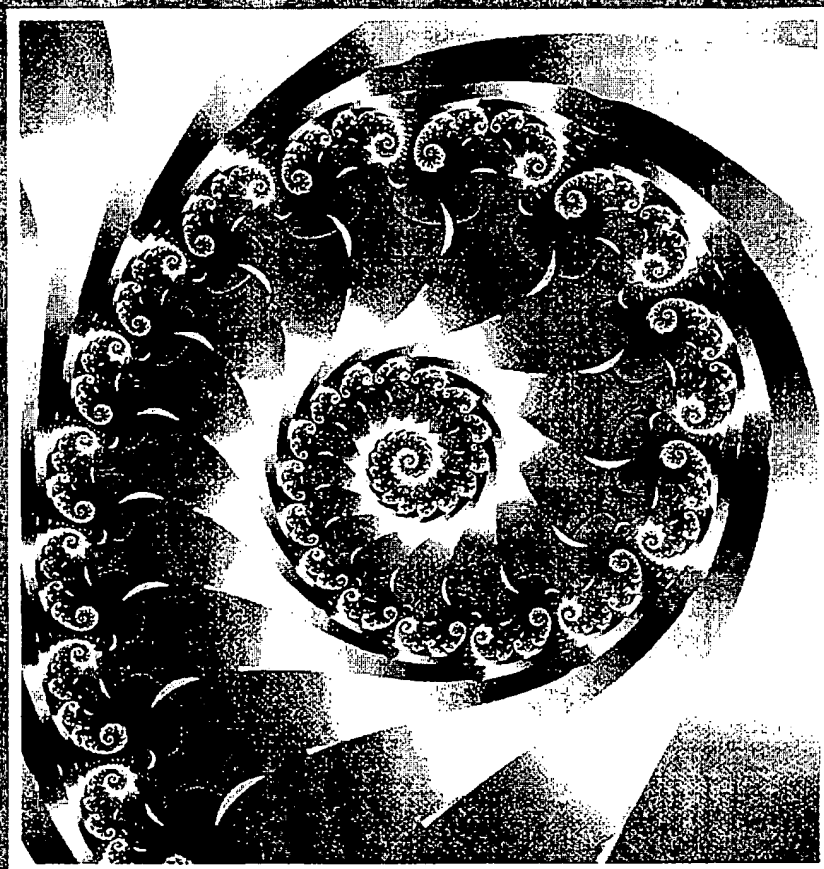
Find  $E(EA)$ .

**Solution**

$$E(EA) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 5 \\ 0 & 3 & 6 \end{bmatrix} \right\}$$

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# LINEAR ALGEBRA



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■ **Dedication:**

This work is dedicated to my mother, Deborah  
my sister, Karen  
and my niece, Chelsea

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3 above.) In general, the matrix  $I_n$ —the  $n \times n$  diagonal matrix with every diagonal entry equal to 1—is called the **identity matrix** of order  $n$  and serves as the multiplicative identity in the set of all  $n \times n$  matrices.

Is there a multiplicative identity in the set of all  $m \times n$  matrices if  $m \neq n$ ? For any matrix  $A$  in  $M_{m \times n}(\mathbb{R})$ , the matrix  $I_m$  is the left identity ( $I_m A = A$ ), and  $I_n$  is the right identity ( $A I_n = A$ ). Thus, unlike the set of  $n \times n$  matrices, the set of nonsquare  $m \times n$  matrices does not possess a unique *two-sided* identity, because  $I_m \neq I_n$  if  $m \neq n$ .

**Example 20:** If  $A$  is a square matrix, then  $A^2$  denotes the product  $AA$ ,  $A^3$  denotes the product  $AAA$ , and so forth. If  $A$  is the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

show that  $A^3 = -A$ .

The calculation

$$A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

shows that  $A^2 = -I$ . Multiplying both sides of this equation by  $A$  yields  $A^3 = -A$ , as desired. [Technical note: It can be shown that in a certain precise sense, the collection of matrices of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

where  $a$  and  $b$  are real numbers, is structurally identical to the

# Elementary Linear Algebra

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Elementary Linear Algebra

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We define  $\delta_{ij}$  to be

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \quad (1.16)$$

This useful symbol is known as the *Kronecker delta*. We shall use it here to construct the  $n$ -th order square matrix  $I_n = [\delta_{ij}]$ . The matrix  $I_n$  is called the  $n$ -th order *identity matrix*. It is a diagonal matrix with 1's in the main diagonal. Usually all matrices under consideration are of the same order, or the order of  $I_n$  is known from context. Thus, usually we shall simply write  $I$  for the identity matrix. If  $A$  is an  $m \times n$  matrix, then  $I_m A = A$  and  $A I_n = A$ . It is easily seen from the definitions that  $\sum_{k=1}^m \delta_{ik} a_{kj} = a_{ij}$ , and  $\sum_{k=1}^n a_{ik} \delta_{kj} = a_{ij}$ . In particular, if

$A$  is a square matrix and  $I$  is an identity of the same order, then  $IA = AI = A$ .

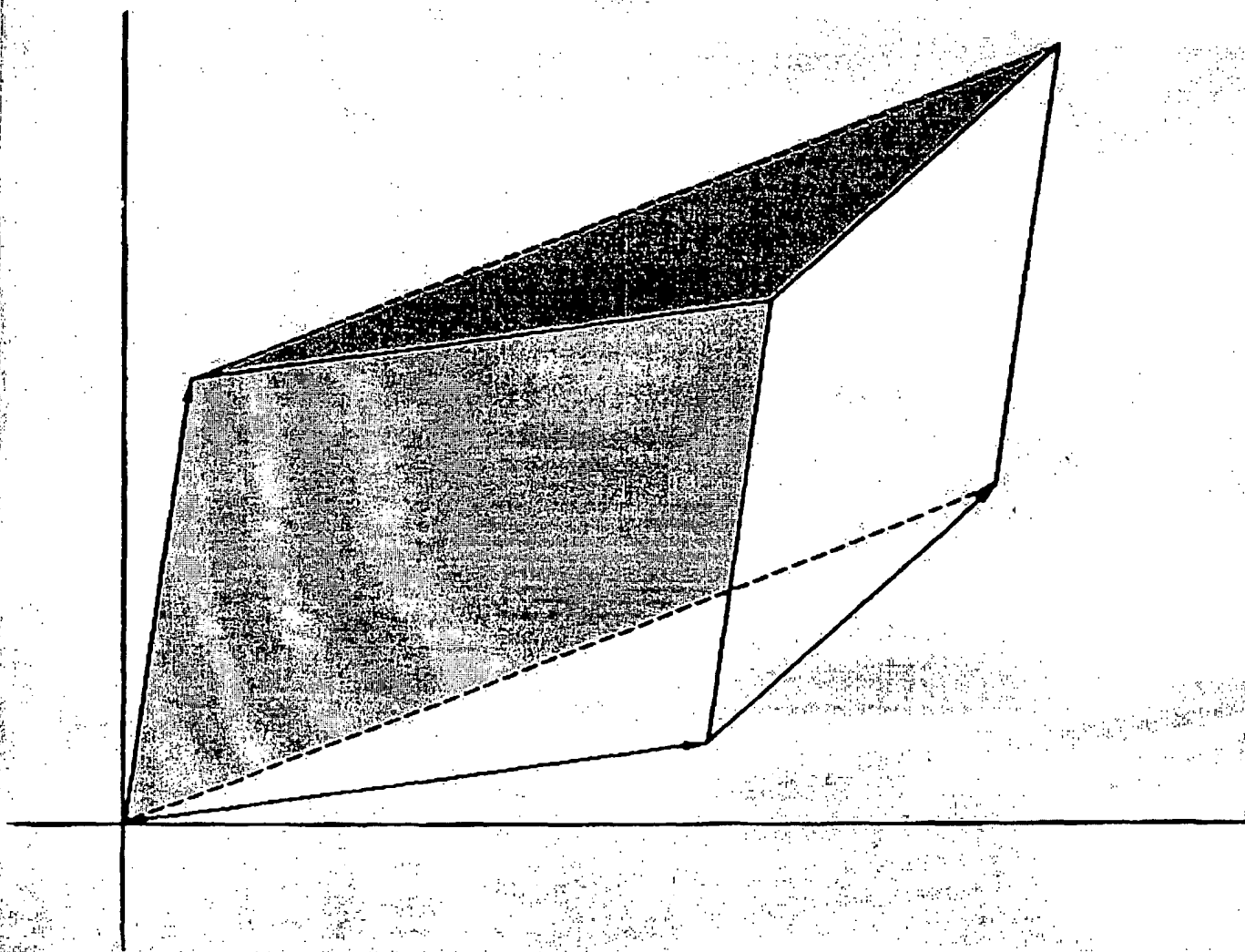
Let  $A$ ,  $B$ , and  $C$  be square matrices of the same order such that  $AB = I$  and  $CA = I$ . Then  $C = CI = C(AB) = (CA)B = IB = B$ .  $B$  is called a *right inverse* of  $A$ , and  $C$  is called a *left inverse* of  $A$ . We have shown that if  $A$  has both a right inverse and a left inverse, then the left inverse and right inverse are identical and that the right inverse is also a left inverse. If  $A$  and  $B$  are square matrices of the same order such that  $AB = BA = I$ , then  $B$  is called an *inverse* of  $A$ . The argument above shows that a matrix cannot have two inverses. We could consider one to be a left inverse and the other to be a right inverse and show they are identical. Thus, we shall speak of *the* inverse of a matrix  $A$  and denote it by  $A^{-1}$ . Since  $AA^{-1} = A^{-1}A = I$ ,  $A$  is also the inverse of  $A^{-1}$ . Later (Section 4-4) we will be able to show that if  $A$  and  $B$  are square matrices of the same order such that  $AB = I$ , then  $BA = I$ .

Matrix notation allows us to express the system of equations (1.3) in extremely compact form. Let  $A$  be the  $m \times n$  matrix whose elements are the coefficients appearing on the left side of (1.3). That is, let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}. \quad (1.17)$$

# MATRICES AND LINEAR ALGEBRA

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In  $F_{n,n}$  we have a special matrix, called the *identity matrix*.

$$I_n = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ \vdots & \vdots & & \vdots \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{bmatrix}$$

defined by

$$e_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

For example,

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$I_n$  has the property that for any  $A$  in  $F_{n,n}$ ,

$$I_n A = A = A I_n.$$

We note in passing that if  $B$  is in  $F_{m,n}$ ,

$$I_m B = B \quad \text{and} \quad B I_n = B,$$

and that for any scalar  $\alpha$ ,

$$(\alpha I_m) B = \alpha B \quad \text{and} \quad B(\alpha I_n) = \alpha B.$$

Recall that for any number  $x \neq 0$ , there is a number  $y = x^{-1}$ , called the *multiplicative inverse* (or reciprocal) of  $x$ , such that

$$xx^{-1} = 1.$$

This situation does not hold in  $F_{n,n}$ , however. For let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

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